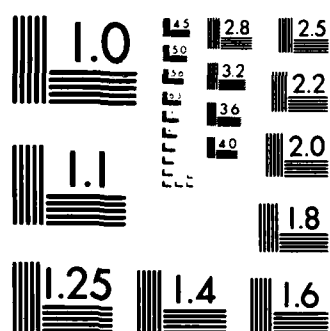END

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FINITE ELEMENT ANALYSIS OF NONLINEAR OSCILLATIONS

AND FLUTTER OF SHELLS OF REVOLUTION

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Finite element methods are described for analysis of nonlinear oscillations of stationary and rotating shells of revolutions. An asymptotic analysis of nonlinear flutter of axisymmetric shells is also presented.

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ACKNOWLEDGMENT

SUMMARY

'In Chapter I, a finite element methodology is described for analysis of steady nonlinear periodic oscillations of axisymmetric shells. The technique is a combination of asymptotic analysis and the finite element method and the results that it yields include the significant effect of nonlinear interaction between the waves that travel in opposite directions along the circumference.

In Chapter II, an extended numerical method is described which is applicable to shells rotating at a constant speed about their geometrical axes.

In the final Chapter III, a general solution for the problem of nonlinear flutter of axisymmetric shells is presented. The method is expected to be useful for developing a numerical scheme for analyzing this class of problems.

CHAPTER I

FINITE ELEMENT ANALYSIS OF NONLINEAR OSCILLATIONS

OF SHELLS OF REVOLUTION

INTRODUCTION

In this chapter a computational methodology is presented for analysis of steady nonlinear harmonic oscillations of stationary shells of revolution. The development is based upon the results which were derived earlier [1] in the context of the more general problem of vibrations of slowly rotating axisymmetric shells. In brief, the results in [1] are in the form of a sequence of linear equations whose solutions can be used for the calculation of the coefficients in the asymptotically exact nonlinear amplitude-frequency equations for the shell. In what follows is presented a finite element method for solution of these linear problems; also described herein are some numerical results which have been obtained from an implementation of the proposed scheme on a computer.

In the class of problems of interest in this chapter, the case of a circular cylindrical shell has been most extensively studied. A historical account of various investigations, which need not be recited here, was given by Evensen [2] about a decade ago. Although even at that time some systematic perturbation analyses of a simply supported circular cylindrical shell were available [3,4], further studies of the simpler problem of a circular ring [1,5-7] clarified a number of issues arising from Evensen's review. It is apparent from this review, and from the results in [1,3,4] that the most interesting feature of nonlinear oscillations of shells of revolution is the interaction between the circumferentially traveling waves that occurs due to the rotational symmetry of the structure. In light of the facts that some proposed finite element based schemes for analysis of nonlinear oscillations of elastic structures even in absence of mode-interaction are not generally valid [8] and that attempts to use the finite element method for vibrations

of axisymmetric shells (by virtue of yielding unacceptable results) have generated much controversy [9-11], the methodology presented in what follows appears to be of some technical interest.

Since the object of the analysis of the type described here is to obtain the relationships between the amplitude and frequency of oscillation of the shell, the same calculations could be done, at least in principle, by using a commercially available finite-element code. Such a computation would entail a series of time-integrations with different set of initial conditions, with the amplitude and frequency obtained from the time-history of motion. This methodology is, however, expected to be quite uneconomical in comparison to the asymptotic method we propose here: the economy evidently results from a systematic elimination of both time and the circumferential coordinate. The efficiency of the scheme becomes more evident in case of the forced vibration problems for which the response of the shell is qualitatively very rich (cf. Fig. 2 of [3]). Although the forced response curve on the basis of the asymptotic analysis can be obtained by solving a small number of nonlinear algebraic equations, it is unlikely that a brute-force time integration scheme can predict all the features of the rather complicated behavior of the shell near a natural frequency as in [3].

ASYMPTOTIC RESULTS

For the sake of easy reference, in this section we present the results for stationary shells obtained from [1]. For our purpose these results have obviously been modified to exclude the terms that arise due to the rotational motion of the shell.

It is convenient to write the equations of motion in the weak form

$$M_{11}(\ddot{q},\delta q) + P_{11}(q,\delta q) + P_{21}(q,\delta q) + P_{31}(q,\delta q) = 0 , \qquad (1)$$

where the notation due to Koiter [12] has been used. Specifically, the array q represents the three displacement components of the shell. The quantity $P_{11}(q,\delta q)$ in (1) is the first variation of the quadratic term in the functional for strain energy, i.e.

$$P_{11}(q,\delta q) = \delta P_2(q) \qquad (2)$$

and, similarly, P_{21} and P_{31} denote quadratic and cubic terms, respectively, in the equilibrium equations. They represent the first variations of homogeneous functionals of degree three and four, respectively, i.e.,

$$P_{21}(q,\delta q) = \delta P_3(q), \quad P_{31}(q,\delta q) = \delta P_4(q) \quad . \qquad (3)$$

As should be obvious, the first term in (1) represents the inertia of the shell in terms of the first variation of a quadratic functional M_2, with $M_2(\dot{q})$ being the kinetic energy of the shell.

The form (1) of the equilibrium equation represents them in a very compact fashion. The form includes both the differential equations and the natural boundary conditions (or algebraic equations for discrete systems) and is valid for any structure. Moreover, for shells of revolution, which are of interest here, each of the functionals in (1) is of the type exemplified by

$$P_2(q) = \int_{s_1}^{s_2} \int_0^{2\pi} r P_2(q) \, d\theta \, ds, \qquad (4)$$

with the components of q depending upon the angular coordinate θ, another spatial coordinate s, and time $\bar{t}$. Evidently, all the components of q must be periodic functions of θ, with a period 2π.

In order to obtain solutions of (1) which are periodic in time with a frequency ω, it is convenient to scale time by ω. In terms of the scaled time coordinate the equilibrium equations become

$$\omega^2 M_{11}(\ddot{q}, \delta q) + P_{11}(q, \delta q) + P_{21}(q, \delta q) + P_{31}(q, \delta q) = 0, \qquad (5)$$

where, now, the superposed dots denote derivative with respect to the non-dimensional time. To (5), one must append the appropriate kinematic conditions on the two ends s_1 and s_2 of the shell, together with the periodicity conditions

$$q(s, \theta, t) = q(s, \theta + 2\pi, t) = q(s, \theta, t + 2\pi). \qquad (6)$$

Equations (5, 6) define a set of nonlinear equations with ω as a parameter; these equations have a trivial solution for all values of ω. The condition for the existence of bifurcation points on this trivial solution branch in the (ω, q) space leads to the classical linear free vibration problem

$$-\omega_0^2 M_{11}(y, \delta q) + P_{11}(y, \delta q) = 0, \qquad (7)$$

where ω_0 is a natural frequency, and y is the associated natural mode of oscillation. Due to rotational symmetry, there are two linearly dependent solutions of (7) for a given natural frequency ω_0, i.e., $y(s, \theta)$ is of the form

$$Y(s) e^{in\theta}, \; Y(s) e^{-in\theta} \; ,$$

with the time dependent motion being a linear combination of the four

solutions

$$Y(s)e^{\pm in\theta}e^{\pm it}.$$

This, of course, is the result based upon the linear theory. The asymptotic

analysis in [1] yields that near a natural frequency ω_0, the solution of (1)

is given by (with c.c. denoting the complex conjugate of the term preceding it)

$$q = (\alpha y e^{it} + c.c.) + (\beta \bar{y} e^{it} + c.c.)$$

$$+ (\alpha^2 z_{11}^{(2)} e^{2it} + c.c.) + \alpha\bar\alpha z_{11}^{(0)}$$

$$+ (\beta^2 z_{22}^{(2)} e^{2it} + c.c.) + \beta\bar\beta z_{22}^{(0)}$$

$$+ (\alpha\beta z_{12}^{(2)} e^{2it} + c.c.) + (\alpha\bar\beta z_{12}^{(0)} + c.c.) + h.o.t., \qquad (8)$$

where h.o.t. represents terms which are at least cubic in the amplitudes α or β or

linear in $(\omega - \omega_0)$. The participating modes $z_{ij}^{(k)}$ in (8) are obtained

from the solution of the linear problems

$$-4\omega_0^2 M_{11}(z_{11}^{(2)}, \delta q) + P_{11}(z_{11}^{(2)}, \delta q) + P_{21}(y, \delta q) = 0 \qquad (9a)$$

$$P_{11}(z_{11}^{(0)}, \delta q) + P_{111}(y, \bar{y}, \delta q) = 0, \qquad (9b)$$

$$-4\omega_0^2 M_{11}(z_{12}^{(2)}, \delta q) + P_{11}(z_{12}^{(2)}, \delta q) + P_{111}(y, \bar{y}, \delta q) = 0 \qquad (9c)$$

$$P_{11}(z_{12}^{(0)}, \delta q) + P_{111}(y, y, \delta q) = 0, \qquad (9d)$$

with

$$z_{22}^{(2)} = \bar{z}_{11}^{(2)}, \quad z_{22}^{(0)} = z_{11}^{(0)}, \qquad (10)$$

and θ-dependence of $z_{ij}^{(k)}$ is of the form

$$z_{11}^{(2)}(s,\theta) = Z_{11}^{(2)}(s)e^{2in\theta}, \quad z_{11}^{(0)}(s,\theta) = Z_{11}^{(0)}(s)$$

$$z_{12}^{(2)}(s,\theta) = Z_{12}^{(2)}(s), \quad z_{12}^{(0)} = Z_{12}^{(0)}(s)e^{2in\theta}.$$

$$(11)$$

Finally, the (complex) amplitudes α and β are related to the frequency ω through the amplitude frequency equations

$$-m(\omega^2-\omega_0^2)\alpha + \gamma_1\alpha^2\bar{\alpha} + \gamma_2\alpha\beta\bar{\beta} = 0 , \tag{12a}$$

$$-m(\omega^2-\omega_0^2)\beta + \gamma_1\beta^2\bar{\beta} + \gamma_2\beta\alpha\bar{\alpha} = 0 , \tag{12b}$$

where

$$m = M_{11}(y,\bar{y}) , \tag{13a}$$

$$\gamma_1 = P_{111}(z_{11}^{(2)}, \bar{y},\bar{y}) + P_{111}(z_{11}^{(0)},y,\bar{y}) + P_{211}(y,\bar{y},\bar{y}) , \tag{13b}$$

$$\gamma_2 = P_{111}(z_{11}^{(0)},y,\bar{y}) + P_{111}(z_{12}^{(2)},y,\bar{y})$$
$$+ P_{111}(z_{12}^{(0)},\bar{y},\bar{y}) + 2P_{211}(y,\bar{y},\bar{y}) . \tag{13c}$$

Thus, the essential effects of nonlinearities on vibration of an axisymmetric shell can be determined by calculating the coefficients γ_1 and γ_2 in (12) by means of (13b,c). As a result, in addition to the solution of the linear free oscillations problem, it is sufficient to obtain the solution of the four linear equations (9). In the next section a finite element method is described which yields discrete approximations to equations (7), (9) and (13), and this approximation allows one to implement the scheme just presented on a digital computer.

DISCRETIZATION

The development in this section is based upon the shell theory
derived by Sanders [13] under the classical Kirchoff's hypothesis and
the small-strains, moderate rotations approximation.

We first list the explicit form of various functionals in (1). If
the shell is assumed to be made of an orthotropic material, with the
principal directions of orthotropy being along the circumferential ($\cdot$)
and meridional (s) coordinates, the strain energy is given by (see Fig. 1)

$$P(q) = \int_{s_1}^{s_2} \int_0^{2\pi} r[p_2(q) + p_3(q) + p_4(q)]d\theta ds$$

$$\equiv P_2(q) + P_3(q) + P_4(q) , \tag{14}$$

where

$$p_2(q) = \frac{1}{2} (C_{11}\epsilon_{ss}^2 + C_{22}\epsilon_{\theta\theta}^2 + 2C_{12}\epsilon_{ss}\epsilon_{\theta\theta} + C_{66}\epsilon_{s\theta}^2)$$

$$+ \frac{1}{2} (D_{11}\kappa_{ss}^2 + D_{22}\kappa_{\theta\theta}^2 + 2D_{12}\kappa_{ss}\kappa_{\theta\theta} + D_{66}\kappa_{s\theta}^2) \tag{15a}$$

$$2p_3(q) = C_{11}\epsilon_{ss}(\beta_s^2 + \beta^2) + C_{22}\epsilon_{\theta\theta}(\beta_\theta^2 + \beta^2)$$

$$+ C_{12}[\epsilon_{ss}(\beta_\theta^2 + \beta^2) + \epsilon_{\theta\theta}(\beta_s^2 + \beta^2)]$$

$$+ C_{66}\epsilon_{s\theta}\beta_s\beta_\theta , \tag{15b}$$

$$p_4(q) = \frac{1}{8} [C_{11}(\beta_s^2 + \beta^2)^2 + C_{22}(\beta_\theta^2 + \beta^2)^2 + 2C_{12}(\beta_s^2+\beta^2)(\beta_\theta^2+\beta^2)$$

$$+ C_{66}\beta_s^2\beta_\theta^2] . \tag{15c}$$

The linear strain components ϵ_{ss}, $\epsilon_{\theta\theta}$, $\epsilon_{\theta s}$ κ_{ss}...etc., together with the
three measures of rotational displacements β_s, β_θ and β are given in terms of

the three translational displacement components of the shell; the relevant equations are given in the appendix. The quantities C_{ij} and D_{ij} in (15) are evidently dependent upon the elastic constants of the shell material and the shell thickness.

The Linear Free Vibration Problem

The discrete approximation of the linear free vibration problem (7) can be obtained in a straightforward manner. The displacements associated with the natural mode of oscillation y are written as

$$y = (U^{(n)}(s), -iV^{(n)}(s), W^{(n)}(s))e^{in\theta}, \tag{16}$$

so that on using the kinematic relations one can obtain the other kinematic variables associated with y in the form

$$(\varepsilon_{ss}, \varepsilon_{\theta\theta}, \varepsilon_{s\theta}) = (E_{ss}^{(n)}, E_{\theta\theta}^{(n)}, -i E_{s\theta}^{(n)})e^{in\theta}, \tag{17a}$$

$$(\beta_s, \beta_\theta, \beta) = (B_s^{(n)}, -i B_\theta^{(n)}, -i B^{(n)})e^{in\theta}, \tag{17b}$$

$$(\kappa_{ss}, \kappa_{\theta\theta}, \kappa_{s\theta}) = (\kappa_{ss}^{(n)}, \kappa_{\theta\theta}^{(n)}, -i \kappa_{s\theta}^{(n)})e^{in\theta}. \tag{17c}$$

The relationship between the quantities $E_{ss}^{(n)}$, $E_{\theta\theta}^{(n)}$,... etc., and displacement variables $U^{(n)}$, $V^{(n)}$ and $W^{(n)}$ has been given in the appendix.

In a manner similar to (16), the virtual displacements δq in (7) are written as

$$(\delta u, \delta v, \delta w) = \sum_{n=0}^{\infty} \{(\delta U^{(n)}, -i\delta V^{(n)}, \delta W^{(n)})e^{in\theta} + c.c.\} \tag{18}$$

with analogous expressions for the associated δE_{ss}, $\delta E_{\theta\theta}$, $\delta E_{s\theta}$, ... etc. Substitution of these expressions into (7) with (14, 15a) yields [I]

$$P_{11}(y, \delta q) = 2\pi \int_{s_1}^{s_2} r[\delta E^{(n)T} C E^{(n)} + \delta \kappa^{(n)T} D \kappa^{(n)}]ds, \tag{19}$$

[I] The symmetric matrix C in (19) has the components C_{ij}, $i, j \leq 2$, with its third column being $(0, 0, C_{66})^T$, cf. (15). The matrix D is defined similarly.

where $\underset{\sim}{E}^{(n)} = (E_{ss}^{(n)}, E_{\theta\theta}^{(n)}, E_{s\theta}^{(n)})^T$, and, similarly, $\delta\underset{\sim}{E}^{(n)}$, $\underset{\sim}{\kappa}^{(n)}$ and $\underset{\sim}{\kappa}^{(n)}$ are arrays with real elements. One can now choose appropriate one-dimensional basis functions for the displacements $\underset{\sim}{U}^{(n)} = (U^{(n)}, V^{(n)}, W^{(n)})^T$ (cf. (16)), i.e.,

$$\underset{\sim}{U}^{(n)}(s) = \underset{\sim}{F}^T(s) \underset{\sim}{y}^{(n)} \tag{20}$$

where $\underset{\sim}{F}$ represents the shape functions and $\underset{\sim}{y}^{(n)}$ is the array of generalized displacements. Finally, with the relations (A.10, 12) written in the more compact form

$$\underset{\sim}{E}^{(n)} = \hat{\underset{\sim}{E}}^{(n)}\underset{\sim}{U}^{(n)}, \quad \underset{\sim}{\kappa}^{(n)} = \hat{\underset{\sim}{\kappa}}^{(n)}\underset{\sim}{U}^{(n)}, \tag{21}$$

and with (20), we obtain

$$P_{11}(y, \delta q) \cong 2 \cdot \delta q^{(n)T}\underset{\sim}{K}^{(n)}\underset{\sim}{y}^{(n)}, \tag{22}$$

where

$$\underset{\sim}{K}^{(n)} = \int_{s_1}^{s_2} [(\hat{\underset{\sim}{E}}^{(n)}\underset{\sim}{F}^T)^T \underset{\sim}{C}(\hat{\underset{\sim}{E}}^{(n)}\underset{\sim}{F}^T)$$
$$+ (\hat{\underset{\sim}{\kappa}}^{(n)}\underset{\sim}{F}^T)^T \underset{\sim}{D}(\hat{\underset{\sim}{\kappa}}^{(n)}\underset{\sim}{F}^T)]ds, \tag{23}$$

is the stiffness matrix associated with a mode of deformation with the circumferential wave number n. The corresponding mass matrix, being obtainable from the functional

$$M_{11}(y, \delta q) = \int_{s_1}^{s_2}\int_0^{2\pi} r\rho(u\delta u + v\delta v + w\delta w)d\theta ds, \tag{24}$$

can be calculated in a similar fashion, and is given by

$$\underset{\sim}{M}^{(n)} = \int_{s_2}^{s_1} r\rho\underset{\sim}{F}\underset{\sim}{F}^T ds, \tag{25}$$

where ρ is the density of the material of the shell. Obviously, both of these matrices are obtained by means of the usual finite element assembly

process. The solution of the generalized eigenvalue problem associated
with these matrices, i.e.,

$$(-\omega_0^2 \underset{\sim}{M}^{(n)} + \underset{\sim}{K}^{(n)})\underset{\sim}{y}^{(n)} = 0, \qquad ||\underset{\sim}{y}^{(n)}|| \neq 0, \qquad (26)$$

provides an approximation to the solution of the problem (7), the linear
free vibration problem.

The Problems for the Participating Modes and the Amplitude-Frequency Equations;

For the solution of the nonlinear problem, it now remains to obtain
discrete analogs of (9a-d) and (13). For this purpose it is convenient
to first consider the forcing terms in (9) (i.e., $P_{21}(y, \delta q), P_{111}(y, \bar{y}, \delta q)$, etc.)
which require the calculation of the first and second variations (Frechet
derivatives) of the functional $P_3(q)$ defined by (14) and (15b). Although
the algebra is tedious, the calculations are conceptually quite straight-
forward. We begin by noting, with equations (3), (14) and (15b), that

$$P_{21}(q, \delta q) = \int_{s_1}^{s_2} \int_0^{2\pi} r p_{21}(q, \delta q) \, d\theta \, ds, \qquad (27)$$

and that

$$
\begin{aligned}
2p_{21}(q, \delta q) = {} & \delta\epsilon_{ss}[C_{11}(\beta_s^2 + \beta^2) + C_{12}(\beta_\theta^2 + \beta^2)] \\
& + \delta\epsilon_{\theta\theta}[C_{12}(\beta_s^2 + \beta^2) + C_{22}(\beta_\theta^2 + \beta^2)] \\
& + \delta\epsilon_{s\theta}[C_{66}\epsilon_s\beta_\theta] \\
& + \delta\beta_s[2C_{11}\epsilon_{ss}\beta_s + 2C_{12}\epsilon_{\theta\theta}\beta_s + C_{66}\epsilon_{s\theta}\beta_\theta] \\
& + \delta\beta_\theta[2C_{12}\epsilon_{ss}\beta_\theta + 2C_{22}\epsilon_{\theta\theta}\beta_\theta + C_{66}\epsilon_{s\theta}\beta_s] \\
& + \delta\beta[2C_{11}\epsilon_{ss}\beta + 2C_{22}\epsilon_{\theta\theta}\beta + 2C_{12}(\epsilon_{ss} + \epsilon_{\theta\theta})\beta]. \qquad (28)
\end{aligned}
$$

In order to evaluate $P_{21}(y, \delta q)$ for (9a), we use the equations (16, 17a,b, 18, 27) and (28). The result is

$$P_{21}(y, \delta q) = 2\pi \int_{s_1}^{s_2} r(\delta E^{(2n)T}a + \delta B^{(2n)T}b)ds, \qquad (29)$$

where

$$a_1 = [C_{11}(B_s^{(n)2} - B^{(n)2}) - C_{12}(B_\theta^{(n)2} + B^{(n)2})]/2, \qquad (30a)$$

$$a_2 = [C_{12}(B_s^{(n)2} - B^{(n)2}) - C_{22}(B_\theta^{(n)2} + B^{(n)2})]/2, \qquad (30b)$$

$$a_3 = C_{66} B_s^{(n)} B_\theta^{(n)}/2, \qquad (30c)$$

$$b_1 = C_{11}E_{ss}^{(n)}B_s^{(n)} + C_{12}E_{\theta\theta}^{(n)}B_s^{(n)} - C_{66}E_{s\theta}^{(n)}B_\theta^{(n)}/2, \qquad (30d)$$

$$b_2 = C_{12}E_{ss}^{(n)}B_\theta^{(n)} + C_{22}E_{\theta\theta}^{(n)}B_\theta^{(n)} + C_{66}E_{s\theta}^{(n)}B_s^{(n)}/2, \qquad (30e)$$

$$b_3 = [C_{11}E_{ss}^{(n)} + C_{22}E_{\theta\theta}^{(n)} + C_{12}(E_{ss}^{(n)} + E_{\theta\theta}^{(n)})]B^{(n)} . \qquad (30f)$$

We recall here that the quantities $E_{ss}^{(n)}$, $B_s^{(n)}$, etc. in (30) are related to the displacements associated with y via equation (A.10, 11).

The discrete analog of (29) is obtained by using (18,20,21). Thus, first the quantities a_i and b_i in (30) are calculated from (A.10, 11) and (18). For example,

$$a_3 = C_{66}\{[1/r_\phi: 0: \frac{-d}{ds}]f^T(s)y^{(n)}\}\{[0: \sin\phi/r: n/r]f^T(s)y^{(n)}\} .$$

From the solution of the homogeneous problem, the quantity a_3 and, similarly, all components of a and b can be calculated for any value of s. Using these functions in (29), one obtains

$$P_{21}(y, \delta q) \simeq 2\pi \ \delta q^{(2n)T}f^{(2)} , \qquad (31)$$

where

$$f^{(2)} = \int_{s_1}^{s_2} r\{[\hat{E}^{(2n)}F^T(s)]^T \underset{\sim}{a}(s) + [\hat{B}^{(2n)}F^T(s)]^T \underset{\sim}{b}(s)\}\,ds \ . \tag{32}$$

In (32) we have used the definition $(21)_1$ and an analogous definition

$$\underset{\sim}{B}^{(i)} = (B_s^{(i)}, \ B_\theta^{(i)}, \ B^{(i)})^T = \hat{\underset{\sim}{B}}^{(i)} \underset{\sim}{U}^{(i)} , \tag{33}$$

with $\hat{B}^{(i)}$ obtained from the equations (A.11).

The remaining forcing term in (9) is $P_{111}(y,\bar{y},\delta q)$ for which the second variation of P_3 needs to be calculated. Since the details are similar to the ones described for computing $P_{21}(y,\delta q)$, we list only the final results. Thus, one obtains

$$P_{111}(y,\bar{y},\delta q) = \int_{s_1}^{s_2} r(\delta \underset{\sim}{E}^{(o)^T} \underset{\sim}{c} + \delta \underset{\sim}{B}^{(o)^T} \underset{\sim}{d})\,ds, \tag{34}$$

where

$$c_1 = C_{11} \ (B_s^{(n)^2} + B^{(n)^2}) + C_{12}(B_\theta^{(n)^2} + B^{(n)^2}), \tag{35a}$$

$$c_2 = C_{12} \ (B_s^{(n)^2} + B^{(n)^2}) + C_{22}(B_\theta^{(n)^2} + B^{(n)^2}), \tag{35b}$$

$$c_3 = 0, \tag{35c}$$

$$d_1 = 2 \ C_{11} E_{ss}^{(n)} B_s^{(n)} + 2 C_{12} E_{\theta\theta}^{(n)} B_s^{(n)} + C_{66} E_{s\theta}^{(n)} B_\theta^{(n)} \ , \tag{35d}$$

$$d_2 = d_3 = 0. \tag{35e}$$

Finally, the discrete analog of (34) turns out to be

$$P_{111}(y,\bar{y},\delta q) \simeq 2\pi \ \delta \underset{\sim}{q}^{(o)^T} \underset{\sim}{f}^{(o)} . \tag{36}$$

with

$$\underset{\sim}{f}^{(o)} = \int_{s_1}^{s_2} r\{ \ [\hat{\underset{\sim}{E}}^{(0)}F^T(s)]^T \underset{\sim}{c} + [\hat{\underset{\sim}{B}}^{(0)}F^T(s)] \underset{\sim}{d}\}\,ds. \tag{37}$$

Our description of the procedure for discretization of (9) is now essentially complete. With the vectors $\underset{\sim}{f}^{(2)}$ and $\underset{\sim}{f}^{(0)}$ defined by (32) and

(37), the approximations to the participating modes $z_{ij}^{(k)}$ in (9) can be obtained by solving the linear systems[I]

$$(-4\omega_o^2 M^{(2n)} + K^{(2n)})Z_{11}^{(2)} + f^{(2)} = 0, \tag{38a}$$

$$K^{(o)}Z_{11}^{(o)} + f^{(o)} = 0, \tag{38b}$$

$$(-4\omega_o^2 M^{(0)} + K^{(0)})Z_{12}^{(2)} + f^{(0)} = 0 , \tag{38c}$$

$$K^{(2n)}Z_{12}^{(0)} + 2f^{(2)} = 0. \tag{38d}$$

In (38) $M^{(o)}$, $K^{(o)}$, $M^{(2n)}$ and $K^{(2n)}$ are defined by (25,23) with n replaced by zero and 2n, respectively, and $Z_{ij}^{(k)}$ are discrete analogs of $Z_{ij}^{(k)}$ in (9).

Once the participating modes have been calculated from (38), one only needs to use (13) to calculate the coefficients in the amplitude-frequency equations (12). All the terms in (13) except $P_{211}(y,\bar{y},\bar{y})$ can be obtained by using equations (31) and (37). Consequently, we proceed to calculate the last term in (13b). For this purpose we write (15c) in the form

$$P_4(q) = \frac{1}{8}[C_{11}(\eta^T\eta)^2 + C_{22}(\xi^T\xi)^2 + 2C_{12}(\eta^T\eta)(\xi^T\xi)$$
$$+ C_{66}\beta_s^2\beta_\theta^2], \tag{39}$$

where

$$\eta = (\beta_s : \beta)^T, \qquad \xi = (\beta_\theta : \beta)^T. \tag{40}$$

The second variation of $P_4(q)$ is then found to be

[I]We note that by definition, $P_{111}(y,y,\delta q) = 2P_{21}(y,\delta q)$ so that the forcing term in (9d) is twice that in (9a).

$$P_{211}(q, q_1, {}^{\cdot}q) = \int_{s_1}^{s_2}\int_0^{2\pi} r p_{211}(q, q_1, {}^{\cdot}q)\,ds\,d\theta \quad , \tag{41}$$

where

$$2p_{211}(q, q_1, {}^{\cdot}q) = C_{11}[2(\eta^T \eta_1)(\eta^T \xi_\eta) + (\eta^T \eta)(\eta_1^T \xi_\eta)]$$

$$+ C_{22}[2(\xi^T \xi_1)(\xi^T \xi) + (\xi^T \xi)(\xi_1^T \xi)]$$

$$+ C_{12}[2(\xi_1^T \xi)(\xi \eta^T \eta) + (\xi^T \xi)(\eta^T \eta_1)$$

$$+ 2(\eta_1^T \eta)(\xi \xi^T \xi) + (\eta^T \eta)(\xi \xi^T \xi_1)]$$

$$+ \frac{1}{2} C_{66}[(\varepsilon_{\theta 1} \varepsilon_s + \delta\varepsilon_\theta \varepsilon_{s1})\varepsilon_s \varepsilon_\theta$$

$$+ (\varepsilon_\theta \varepsilon_s + \varepsilon_\theta \varepsilon_s)(\varepsilon_s \varepsilon_{\theta 1} + \varepsilon_{s1}\varepsilon_\theta)], \tag{42}$$

and the kinematic quantities with subscript 1 are associated with displacements corresponding to q_1. We now use (17b) and (40-42) to obtain

$$P_{211}(v, \bar{y}, \bar{y}) = 2\pi \int_{s_1}^{s_2} p\, r\, ds \equiv 2\pi\gamma \tag{43}$$

where

$$p = \frac{1}{2}[C_{11}(2g_1^2 + g_2^2) + C_{22}(2h_1^2 + h_2^2)]$$

$$+ C_{12}(2g_1 h_1 + g_2 h_2) + \frac{1}{2} C_{66}(B_\theta^{(n)} B_s^{(n)})^2 \tag{44}$$

and

$$g_1 = B_s^{(n)2} + B^{(n)2}, \quad g_2 = B_s^{(n)2} - B^{(n)2}, \tag{44a}$$

$$h_1 = (B_\theta^{(n)2} + B^{(n)2}), \quad h_2 = -h_1. \tag{44b}$$

Thus once the free vibration problem has been solved, the left hand side of (43) can be evaluated by using the equations (20), (A.11) and (43). Finally, from

these results one obtains the discrete form of (13) to be

$$m/2^- = \underset{\sim}{v}^{(n)T} \underset{\sim}{M}^{(n)} \underset{\sim}{v}^{(n)} \, , \tag{45a}$$

$$\Omega_1/2^- = 2\underset{\sim}{f}^{(2)T} \underset{\sim}{z}_{11}^{(2)} + \underset{\sim}{f}^{(o)T} \underset{\sim}{z}_{11}^{(o)} + \gamma \, , \tag{45b}$$

$$\Omega_2/2^- = \underset{\sim}{f}^{(o)T} \underset{\sim}{z}_{11}^{(o)} + \underset{\sim}{f}^{(0)T} \underset{\sim}{z}_{12}^{(2)}$$

$$+ 2\underset{\sim}{f}^{(2)T} \underset{\sim}{z}_{12}^{(o)} + 2\gamma \, . \tag{45c}$$

In summary, the computations proceed in the following manner. First, for a given circumferential wave number n, the stiffness and mass matrices are obtained by using (23,25). The solution of the generalized eigenvalue problem (26) yields a natural mode of oscillation and the associated frequency. The natural mode is used for calculation of the forcing terms $\underset{\sim}{f}^{(2)}$ and $\underset{\sim}{f}^{(0)}$ via (32,37). Then the mass and stiffness matrices corresponding to circumferential wave numbers 2n and zero are assembled so that the participating modes can be calculated from (38). Finally with the scalar γ calculated by means of (43,44), equations (45a-c) are used to compute the desired coefficients the amplitude-frequency equations (12). The results from a computer implementation of the procedure just described are presented in the next section.

NUMERICAL RESULTS

In the program that has been developed on the IBM 4341 at the Yale Computer Center, one dimensional Hermite cubics were used as the shape functions for all the three displacement components, together with four point Gaussian quadrature rule for numerical integration. All the computations

were done in double precision. For the purpose of comparing the results
with those obtained previously, the scalars

$$P_1 = \gamma_1 / (8m\omega_o^2), \tag{46a}$$

$$P_2 = (\gamma_2 - \gamma_1) / (16m\omega_o^2), \tag{46b}$$

were computed. The scalars occur in a set of amplitude-frequency equations
equivalent to (12), to wit,

$$-(\Omega^2 - 1)a + (P_1 + P_2)a^3 + (P_1 + P_2\cos2\psi)ab^2 = 0, \tag{47a}$$

$$ab^2\sin2\psi \ P_2 = 0, \tag{47b}$$

$$-(\Omega^2 - 1)b + (P_1 + P_2)b^3 + (P_1 + P_2\cos2\psi)ba^2 = 0, \tag{48a}$$

$$-ba^2\sin2\psi \ P_2 = 0, \quad \Omega \equiv \omega/\omega_o. \tag{48b}$$

These equations are obtained by setting

$$\alpha = (a + ibe^{i\psi})/4, \tag{49a}$$

$$\beta = (a - ibe^{i\psi})/4, \tag{49b}$$

in (12), adding (12a) to (12b) and subtracting (12b) from (12a), using the
definitions (46), and satisfying both the real and the imaginary parts of
the resulting equations. Evidently, equations (48) are equivalent to those
derived by Ginsberg [3] for the special case of a circular cylindrical shell.

In Table 1 are presented the results of a convergence study for a simply
supported circular cylindrical shell analyzed in [3]. Only half of the
shell was analyzed, with symmetry conditions imposed at $s = 0$, ($w' = v' = u = 0$)
and the simple support conditions ($v = w = 0$) at $s/R = L/2R$, with L being the

length of the shell and R its radius.   It is evident from Table 1 that

the convergence of the various coefficients with increase in the number

of elements is rapid and monotonic.

In Table 2, the computed results have been compared with those

in   [3]. Although one of the reasons for the differences could be the

different shell theory used in [3], the rather substantial discrepancy

cannot be explained solely on this basis. (We note here that ultimately

Ginsberg's analysis is also a numerical scheme involving summation of a

convergent series). It may also be noted here that a direct numerical

counterpart of Ginsberg's analysis would entail the rather uneconomical

computation of a large number (theoretically all) of the natural modes of

oscillation and the associated frequencies.

In Table 3 we have presented some results obtained for a simply

supported frustum (conical shell). Specifically, only the circumferential

and transverse displacements of the shell were constrained to vanish at its

two ends. For the semi-vertex angle $\alpha = \pi/2$, the frustum degenerates to

an annular plate, and our results indicate that for this case the non-

linearity is of the hardening type for both the circumferentially traveling

waves and standing waves. [I]

[I] It may be noted here that from (47-48) the following amplitude frequency

equations can be derived:

(a) $b = 0$, $(\Omega^2 - 1) = (P_1 + P_2)a^2$; Standing Waves,

(b) $a = b, \psi = 0$, $(\Omega^2 - 1) = 2P_1 a^2$; Traveling Waves.

The character of these solutions can be readily verified by using (49) in

the leading part of (8).

CONCLUDING REMARKS

A numerical method based on asymptotic results has been developed
for analysis of steady nonlinear harmonic oscillations of shells of re-
volution. Some results have been described for circular cylindrical
and conical shells. As discussed in [1], once the coefficients in the
amplitude frequency equations have been obtained for the free vibration
problem, the problem of forced vibration can be reduced to solution of
two complex nonlinear equations. Thus our results can be directly used for
the calculation of the response of harmonically excited shells of revolution.
Finally, further extension of the numerical method to include the effect of
steady rotation of the shell is the subject of the next chapter.

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APPENDIX

A. Strain-Displacement and Rotation-Translational Displacement Relations

(See Fig. 1 for definitions of r, r_ϕ and ϕ).

Membrane Strains

$$\varepsilon_{ss} = u_{,s} + w/r_\phi \tag{A.1}$$

$$\varepsilon_{\theta\theta} = (v_{,\theta}/r + u \cos\phi/r + w \sin\phi/r) \tag{A.2}$$

$$\varepsilon_{s\theta} = (v_{,s} + u_{,\theta}/r - v \cos\phi/r)/2 \tag{A.3}$$

Rotations

$$\beta_s = -w_{,s} + u/r_\phi \tag{A.4}$$

$$\beta_\theta = (-w_{,\theta} + v \sin\phi)/r \tag{A.5}$$

$$\beta = (v_{,s} + v \cos\phi/r - u_{,\theta}/r)/2 \tag{A.6}$$

Bending Strains

$$\kappa_{ss} = -w_{,ss} + u_{,s}/r_\phi - u \, r_{\phi,s}/r_\phi^2 \tag{A.7}$$

$$\kappa_{\theta\theta} = (\beta_{\theta,\theta} + \beta_s \cos\phi)/r \tag{A.8}$$

$$\kappa_{s\theta} = [\beta_{\theta,s} + \beta_{s,\theta}/r - \beta_\theta \cos\phi/r + (\sin\phi/r - 1/r_\phi)\beta]/2 \tag{A.9}$$

B. Kinematic Relations for the Quantities in (17) for Deformation Associated with Circumferential Wave Number n

$$\begin{pmatrix} E_{ss}^{(n)} \\[2mm] E_{\theta\theta}^{(n)} \\[2mm] E_{s\theta}^{(n)} \end{pmatrix} = \begin{pmatrix} d/ds & 0 & 1/r_\phi \\[2mm] \cos\phi/r & n/r & \sin\phi/r \\[2mm] -n/2r & \frac{1}{2}(d/ds - \cos\phi/r) & 0 \end{pmatrix} \begin{pmatrix} U^{(n)} \\[2mm] V^{(n)} \\[2mm] W^{(n)} \end{pmatrix}$$

$$\equiv \underset{\sim}{\mathcal{L}}^{(n)} \underset{\sim}{U}^{(n)} . \tag{A.10}$$

$$
\begin{pmatrix} B_s^{(n)} \\ B_r^{(n)} \\ B^{(n)} \end{pmatrix} = \begin{pmatrix} 1/r & 0 & -d/ds \\ 0 & \sin\phi/r & n/r \\ n/2r & \frac{1}{2}(\cos\phi/r + d/ds) & 0 \end{pmatrix} \begin{pmatrix} U^{(n)} \\ V^{(n)} \\ W^{(n)} \end{pmatrix}
$$

$$
\equiv \hat{B}^{(n)} U^{(n)} . \tag{A.11}
$$

$$
\begin{pmatrix} \kappa_{ss}^{(n)} \\ \kappa_{\phi\phi}^{(n)} \\ \kappa_{s\phi}^{(n)} \end{pmatrix} = \begin{pmatrix} d/ds & 0 & 0 \\ \cos\phi/r & n/r & 0 \\ n/2r & \frac{1}{2}(d/ds - \cos\phi/r) & \frac{1}{2}(\sin\phi/r - 1/r_\phi) \end{pmatrix} \hat{B}U^{(n)}
$$

$$
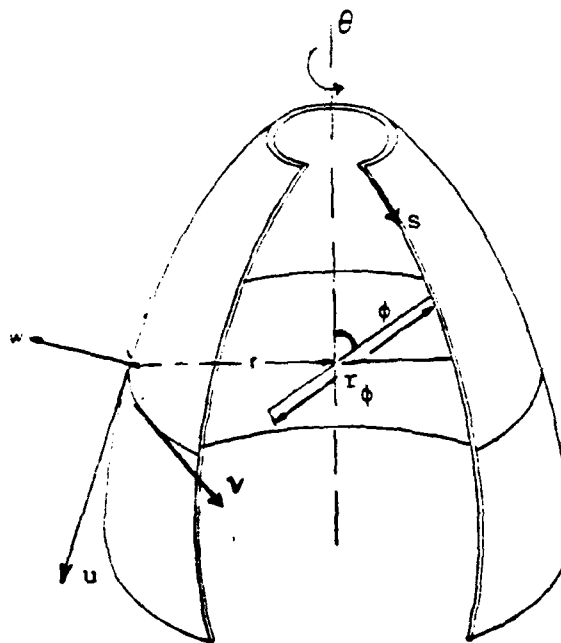\equiv \hat{\kappa}^{(n)} U^{(n)} . \tag{A.12}
$$

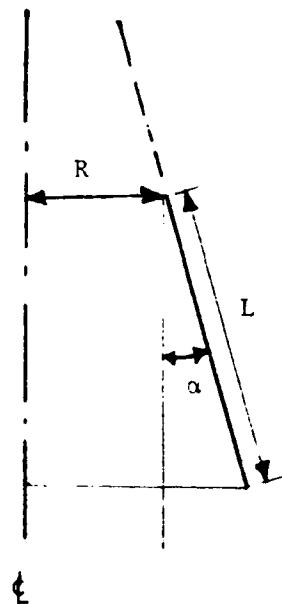Fig. 1 Geometry and Coordinate System.

Fig. 2   Geometry of the Frustum Analyzed.

Table 1

Convergence Study for a Circular Cylindrical Shell

of Isotropic Material, Poisson's Ratio $\nu = .3$.

L/R[(1)]	n[(2)]	No. of Elements	ω_o[(3)]	$\tilde{w}_{max}$[(4)]	P_1[(5)]	P_2[(5)]
1	2	5	.6531	.9576	−5.361	−74.17
		10	.6531	.9570	−5.406	−74.21
		20	.6531	.9570	−5.410	−74.22
	8	5	.2436	.9914	−565.6	−1271.3
		10	.2436	.9914	−566.7	−1271.4
		20	.2436	.9914	−566.7	−1271.4
4	2	5	.1130	.8836	−1.489	−61.70
		10	.1130	.8836	−1.490	−61.70
		20	.1130	.8836	−1.490	−61.70
	8	5	.1824	.9921	−6.805	−822.4
		10	.1824	.9921	−6.809	−822.5
		20	.1824	.9921	−6.810	−822.5

1. L/R = Length to Radius Ratio.

2. n: Circumferential Wave Number.

3. $\omega_o = [(1-\nu^2)\rho R^2/E]^{1/2}\bar{\omega}_o$ where $\bar{\omega}_o$ is the lowest natural frequency associated with n and E is the Young's Modulus.

4. $\tilde{w}_{max} = w_{max}/R$, where w_{max} is the maximum normal displacement associated with the normal mode, normalized such that $(w_{max}^2 + v_{max}^2 + u_{max}^2)/R^2 = 1$, as in [3].

5. Coefficients in amplitude frequency equations (47, 48).

Table 2

A Comparison of Numerical Results with those in [3].

(Parameters and Notations are same as in Table 1.)

L/R	n		$\tilde{\gamma}_o$	$\tilde{w}_{max}$	P_1	P_2
1	2	New Results	.6531	.9576	−5.410	−74.22
		Results from [3]	.6531	.9576	−24.62	−96.00
	8	New Results	.2436	.9914	−566.7	−1271.4
		Results from [3]	.2437	.9914	−1542	−1771
4	2	New Results	.1130	.8836	−1.490	−61.70
		Results from [3]	.1130	.8836	−6.35	−68.15
	8	New Results	.1824	.9921	−6.810	−822.5
		Results from [3]	.1824	.9922	−109.2	−873.1

Table 3

Coefficients in the Amplitude Frequency Equation (47)

for a Simply Supported Frustum.

ι (Degrees)	ω_0	P_1	P_2
15	0.5991	-6.49	-57.71
30	0.4983	$+1.70$	-37.24
45	0.3753	2.85	133.85
60	0.2511	10.43	-97.08
75	0.1303	32.56	-311.16
90 (Annular Plate)	0.0340	469.14	233.60

[$\nu = 0.3$, $R/h = 100$, $L/R = 1$, (see Fig. 2), $n = 2$. Notation is the same as in Table 1.]

CHAPTER II

FINITE ELEMENT ANALYSIS OF NONLINEAR OSCILLATIONS

OF ROTATING AXISYMMETRIC SHELLS

INTRODUCTION

In this chapter we describe a finite-element methodology for analysis of steady nonlinear harmonic oscillations of rotating axisymmetric shells. As in Chapter I, the methodology is based on the asymptotic results derived earlier [1], and it yields an asymptotically exact nonlinear relationship between the amplitude of oscillation of the shell and its frequency.

The main influence of rotation on axisymmetric shells is that it destroys the reflection symmetry with respect to the circumferential direction. As a result, the two waves that travel in opposite directions along the circumference are expected to exhibit different behavior, both in the linear and in the nonlinear range. It is for the assessment of such differences between the two circumferentially traveling waves that the numerical method developed in this chapter can be utilized.

Although in mechanics literature there is a large number of accounts of analytical, numerical and experimental studies on nonlinear oscillations of stationary shells, the author's analysis [1] appears to be the only one in which explicit results have been obtained on the oscillations of a rotating shell type of structure. Due to the analytical intractability of the general problem, the study in [1] was restricted to the simple problem of a ring: evidently the finite-element method described here can be utilized for more general problems.

FORMULATION

As in Chapter I, we use the Sanders' strain-displacement relations under the small-strains, moderate rotations approximation. The strain energy of the shell is then given by the functional $P(q)$ defined by (I.14, 15)[I]. In terms of this functional, the equilibrium equation of the shell in a co-ordinate frame rotating with the shell can be written as

$$\delta P(q) + \int_{s_1}^{s_2} \int_{c}^{2\pi} c(a_s \delta u + a_\theta \delta v + a_r \delta w) r d\theta ds. \tag{1}$$

In this equation, ρ is the mass per unit area of the shell and the quantities

$$a_s = \ddot{u} - 2\Omega\dot{v} \cos\phi - \Omega^2 \cos\phi(u \cos\phi + w \sin\phi) - \Omega^2 r \cos\phi, \tag{2a}$$

$$a_\theta = \ddot{v} + 2\Omega\dot{u} \cos\phi + 2\Omega\dot{w} \sin\phi - \Omega^2 v, \tag{2b}$$

$$a_r = \ddot{w} - 2\Omega\dot{v} \sin\phi - \Omega^2 \sin\phi(u \cos\phi + w \sin\phi) - \Omega^2 r \sin\phi \tag{2c}$$

are the acceleration components of the shell in the rotating frame, see Figure 1. Evidently, the quantity Ω in (2) is the speed of steady rotation of the shell about its geometrical axis.

The last terms of the equations (2a,c) lead to an effective centrifugal force on the shell, so that prior to analysis of oscillations one has to calculate the steady prestressed state about which the vibrations occur. This prestressing due to steady rotation, denoted by q^o, is governed by the nonlinear equations obtained by dropping from (1,2) the terms that contain time derivatives of displacements; to wit:

[I]
In this chapter, equation numbers beginning with Roman I refer to the corresponding equations in Chapter I.

$$\ddot{P}(q^O) + \ddot{}^2 R_{11}(q^O, \ddot{}q) + \ddot{}^2 F_{11}(f, \ddot{}q) = 0, \tag{3}$$

where

$$R_{11}(q, q) \equiv \int_{s_1}^{s_2} \int_0^{2\pi} \left\{ \begin{array}{l} \cos(u\cos + w\sin)\delta u \\ + v\delta v + \sin(u\cos + w\sin)\delta w \end{array} \right\} r d\theta ds, \tag{4a}$$

$$F_{11}(f, c) = - \int_{s_1}^{s_2} \int_0^{2\pi} (r\cos\delta u + r\sin\delta w) r d\theta ds. \tag{4b}$$

The forcing term (4b) in the equation for prestress due to rotation is obviously θ-independent: the prestress state is, therefore, axisymmetric, as expected.

For a fixed value of Ω, the solution of discrete analog of (3) can be obtained by the usual Newton-Raphson method for solution of nonlinear algebraic equations. We write the solution of (1) in the form

$$q = q^o + \tilde{q} \tag{5}$$

with $\tilde{q}$ representing the oscillatory motion of the shell superimposed on the steady deformation. On substitution of (5) in (1,2), together with the use of the fact that q^o satisfies (3), we obtain the homogeneous equation for the free nonlinear oscillations of the shell. This equation can be written as

$$M_{11}(\ddot{\tilde{q}}, \delta q) + G_{11}(\dot{\tilde{q}}, \delta q) + V_{11}(\tilde{q}, \delta q) + V_{21}(\tilde{q}, \delta q) + V_{31}(\tilde{q}, \delta q) = 0 \tag{6}$$

where

$$M_{11}(\ddot{\tilde{q}}, \delta q) = \int_{s_1}^{s_2} \int_0^{2\pi} (\ddot{\tilde{u}}\delta u + \ddot{\tilde{v}}\delta v + \ddot{\tilde{w}}\delta w) r d\theta ds, \tag{7a}$$

$$G_{11}(\tilde{q}, \delta q) = 2\Omega \int_{s_1}^{s_2} \int_0^{2\pi} \left\{ \begin{array}{l} (-\dot{v}\delta u + \dot{u}\delta v)\cos\phi \\ +(\dot{w}\delta v - \dot{v}\delta w)\sin\phi \end{array} \right\} r d\theta ds, \tag{7b}$$

$$V_{11}(\tilde{q},\varsigma q) = P_{11}(\tilde{q},\varsigma q) + P_{111}(q^o,\tilde{q},\delta q) + P_{211}(q^o,\tilde{q},\varsigma q) + \varsigma^2 R_{11}(\tilde{q},\varsigma q), \qquad (7c)$$

$$V_{21}(\tilde{q},\varsigma q) = P_{21}(\tilde{q},\varsigma q) + P_{211}(q, q^o,\delta q), \qquad (7d)$$

$$V_{31}(\tilde{q},\varsigma q) = P_{31}(\tilde{q},\delta q). \qquad (7e)$$

Here, as usual, $P_{11}(q,\varsigma q)$ denotes the first variation of the quadratic part of the strain energy functional; P_{21} and P_{31} are defined similarly. We note here that the bilinear functional G_{11} in (7b) is antisymmetric with respect to the argument, i.e., $G_{11}(q_1, q_2) = - G_{11}(q_2, q_1)$, as can be readily verified.

For subsequent discussion it is convenient to drop the over-tilda from $\tilde{q}$ and to scale time by using the (yet unknown) frequency ω of periodic vibration of the shell. The equation (6) is then modified to

$$\omega^2 M_{11}(\ddot{q},\delta q) + \omega\, G_{11}(\dot{q},\delta q) + V_{11}(q,\delta q) + V_{21}(q,\delta q) + V_{31}(q,\delta q) = 0, \qquad (8)$$

where, now, dot represents derivative with respect to the scaled time coordinate. With this scaling, the object of our analysis reduces to obtaining solution of (8) whose time dependence is periodic with a period 2π.

An asymptotic solution of the problem has been presented in [1]. Here we summarize the main results.

First one calculates a natural frequency ω_o and the corresponding natural mode of oscillation, y, from the quadratic eigenvalue problem

$$-\omega_o^2 M_{11}(y,\delta q) + i\omega_o\, G_{11}(y,\delta q) + V_{11}(y,\delta q) = 0 , \qquad (9)$$

to which must be appended an appropriate normalization condition for y. If for a given natural frequency ω_o, the associated mode of oscillation is unique up to a scalar multiplier, the results in [1] indicate that (8) admits solutions of the form

$$q = (\alpha y e^{it} + c.c.) + (\alpha^2 z^{(2)} e^{2it} + c.c.) + \alpha\bar\alpha z^{(o)} + O(\alpha^3, \alpha(\omega-\omega_o)) , \qquad (10)$$

where   c.c. represents the complex conjugate of the term preceding it.
In (10), the "participating modes" $z^{(2)}$ and $z^{(0)}$ are obtained from the
linear time-independent problems

$$-4\omega_o^2 M_{11}(z^{(2)}, \zeta q) + 2i\omega_o G_{11}(z^{(2)}, \zeta q) + V_{11}(z^{(2)}, \zeta q) + V_{21}(y, \zeta q) = 0, \qquad (11)$$

$$V_{11}(z^{(o)}, \zeta q) + V_{111}(y, \bar y, \zeta q) = 0 . \qquad (12)$$

The complex amplitude α and the frequency ω are related by the nonlinear
amplitude frequency equation

$$-(\omega^2 - \omega_o^2)m\alpha + (\omega - \omega_o)q\alpha + \gamma_1 \alpha^2\bar\alpha = 0, \qquad (13)$$

where the nonlinearity coefficient γ_1, and the other scalars are obtained from

$$m = M_{11}(y, \bar y), \quad g = i G_{11}(y, \bar y), \qquad (14)$$

$$\gamma_1 = 2P_{21}(y, z^{(2)}) + P_{111}(y_1\bar y, z^{(o)}) + P_{211}(y, \bar y, \bar y). \qquad (15)$$

The main object of our calculations, of course, is to obtain the
quantities m, g and γ_1 in (13) so that at least an asymptotic relationship
between the amplitude and frequency is explicitly available. For this purpose,
as we have discussed in this section, one needs the computation of (a) the
steady prestress due to rotation, (b) a natural frequency and the associated
mode of oscillation and (c) the participating modes. Following section is
devoted to some aspects of these computations.

THE DISCRETE ANALOGS

Once the variations of the various functionals needed in (9,11,12,14,15) have been calculated, standard finite element techniques can be used to obtain the discrete approximations of these equations. Since the details are similar to those given in Chapter I, here we only outline some of the main results.

On discretizations the equation (3) for prestressing due to steady rotation becomes

$$h(q^{(o)}) \equiv K^{(o)}q^{(o)} + \frac{1}{2}N1(q^{(o)})q^{(o)} + \frac{1}{3}N2(q^{(o)})q^{(o)}$$

$$+ \Omega^2 R^{(o)}q^{(o)} + \Omega^2 f = 0 \tag{16}$$

where $K^{(o)}$ is the constant stiffness matrix, with $N1$ and $N2$ being symmetric matrices whose elements are linearly and quadratically dependent, respectively, on the argument $q^{(o)}$, the vector of generalized nodal displacements. Equation (16) can be solved by the Newton-Raphson scheme ($i = 1,2,\ldots\ell$):

$$q^{(o)1} = 0, \tag{17a}$$

$$\{K^{(o)} + N1(q^{(o)i}) + N2(q^{(o)i})$$

$$+ \Omega^2 R^{(o)}\} \Delta q^i + h(q^{(o)i}) = 0, \tag{17b}$$

$$q^{(o)i+1} = q^{(o)i} + \Delta q^i, \tag{17c}$$

with ℓ determined by means of a suitable convergence test.

After the prestressed state due to rotation has been obtained, one can solve the discrete analog of the linear free vibration problem (9) which turns out to be of the form

$$[-\omega_o^2 \underset{\sim}{M}^{(n)} + \omega_o \underset{\sim}{G}^{(n)} + \underset{\sim}{K}^{(n)}] \underset{\sim}{y}^{(n)} = 0, \qquad (18a)$$

$$\underset{\sim}{y}^{(n)T} \underset{\sim}{y}^{(n)} = 1, \qquad (18b)$$

where n is the circumferential wave number, and the symmetric matrices $\underset{\sim}{M}^{(n)}$, $\underset{\sim}{G}^{(n)}$ and $\underset{\sim}{K}^{(n)}$ represent the mass, the Coriolis acceleration[1] and the stiffness, respectively.

Although some attention has been given in the literature to the development of special numerical methods for solving quadratic eigenvalue problem of the type (18), we use a simple Newton-Raphson scheme for (18) as well; to wit:

i = 1,2,..., till convergence,

$$\underset{\sim}{A}^i \equiv \begin{bmatrix} -(\omega_o{}^i)^2 \underset{\sim}{M}^{(n)} + \omega_o{}^i \underset{\sim}{G}^{(n)} + \underset{\sim}{K}^{(n)} & (-2\omega_o{}^i \underset{\sim}{M}^{(n)} + \underset{\sim}{G}^{(n)}) \underset{\sim}{y}^{(n)i} \\ \\ 2 \underset{\sim}{y}_n{}^{iT} & 0 \end{bmatrix}, (19a)$$

$$\underset{\sim}{g}^i = \begin{pmatrix} (-\omega_o^2 \underset{\sim}{M}^{(n)} + \omega_o \underset{\sim}{G}^{(n)} + \underset{\sim}{K}^{(n)}) \underset{\sim}{y}^{(n)i} \\ \\ \underset{\sim}{y}^{(n)iT} \underset{\sim}{y}^{(n)i} - 1 \end{pmatrix}, \qquad (19b)$$

---

[1] We note that if we write the displacement components in the form (I.16) and use the definition (7b), the functional iG_{11} in (9) becomes a <u>real symmetric</u>, bilinear functional.

$$\underset{\sim}{A}^i \begin{pmatrix} \Delta y^{(n)i} \\ \\ \Delta \omega_o^{\,i} \end{pmatrix} + \underset{\sim}{g}^i = \underset{\sim}{0}, \tag{19c}$$

$$\underset{\sim}{y}^{(n)i+1} = \underset{\sim}{y}^{(n)i} + \underset{\sim}{\Delta y}^{(n)i}, \tag{19d}$$

$$\omega_o^{\,i+1} = \omega_o^{\,i} + \Delta \omega_o^{\,i}. \tag{19e}$$

Since the calculations can be done for successively increasing values of the speed of rotation Ω and since for $\Omega = 0$ the quadratic eigenvalue problem reduces to the standard algebraic eigenvalue problem due to vanishing of $\underset{\sim}{G}^{(n)}$, the starting iterate for the scheme (19) for a fixed Ω can be taken to be the solution obtained for a previous value of Ω. Some more sophisticated extrapolation schemes could obviously be devised for this purpose, but we have found this method to be sufficient for our purpose.

The calculation of participating modes by solving the linear problems (11,12) is a relatively simple task. The discrete analogs of these equations turn out to be of the type

$$[-4\omega_o^2 \underset{\sim}{M}^{(2n)} + 2\omega_o \underset{\sim}{G}^{(2n)} + \underset{\sim}{K}^{(2n)}]\underset{\sim}{z}^{(2n)} + \underset{\sim}{f}^{(2n)} = \underset{\sim}{0}, \tag{20a}$$

$$\underset{\sim}{K}^{(o)}\underset{\sim}{z}^{(o)} + \underset{\sim}{f}^{(o)} = \underset{\sim}{0} \tag{20b}$$

which are linear algebraic equations for the quantities $\underset{\sim}{z}^{(o)}$ and $\underset{\sim}{z}^{(2n)}$ and therefore can be readily solved by, say, Gaussian elimination.

Finally, with the natural modes of oscillation and the participating modes having been calculated according to the schemes just mentioned, the coefficients in the amplitude frequency equations (13) are obtained from

$$m = \underset{\sim}{y}^{(n)^T} \underset{\sim}{M}^{(n)} \underset{\sim}{y}^{(n)}, \tag{21a}$$

$$g = \underset{\sim}{y}^{(n)^T} \underset{\sim}{G}^{(n)} \underset{\sim}{y}^{(n)}, \tag{21b}$$

$$\gamma_1 = \underset{\sim}{z}^{(2n)^T} \underset{\sim}{f}^{(2n)} + \underset{\sim}{z}^{(o)} f^{(o)} + \gamma, \tag{21c}$$

where the last term is calculated as in (I.43, 44).

NUMERICAL RESULTS

The algorithm described in the previous sections has been implemented on the IBM 4341 at the Yale Computer Center. The program employs one-dimensional Hermite cubics for shape functions for all the three displacement components together with four-point Gaussian quadrature scheme for integration.

In Table 1 are presented the results for a simply supported circular cylindrical shell. All the computations were done by using ten elements for half-length of the shell with symmetry conditions imposed at the mid-length. It is evident that due to rotation there is further "softening" of the nonlinear response. However, it appears from the results that in the range of rotational speeds considered, the effect of rotation on the non-linearity coefficient γ_1 is similar for both the forward traveling wave and the backward traveling wave. Similar trends can be observed in Table 2, in which the results of a similar calculation for a circular shell clamped at both ends have been presented.

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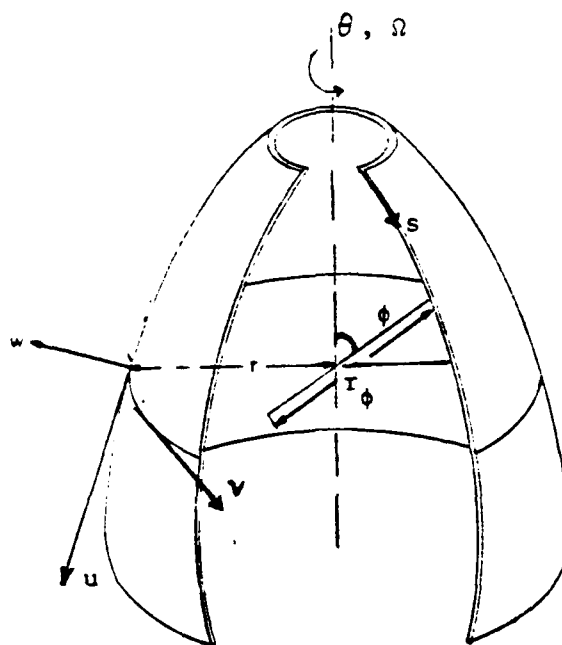   Mechanics, Vol. 48, 1982, p. 627-634.

Fig. 1 Geometry and Coordinate System.

Table 1

Coefficients in the Amplitude Frequency Equation (13)

for a Simply Supported Circular Cylindrical Shell[I]

(L/R = 1, ν = 0.3, R/h = 100, n = 8)

Ω/ω_s[I]	Type of Wave[III]	ω	m	g	γ_1
0	F,B	0.243	0.250	0	– 67.1
0.05	F	0.245	0.250	– 0.76 x 10^{-3}	– 67.9
0.05	B	0.248	0.250	+ 0.76 x 10^{-3}	– 67.9
0.10	F	0.251	0.248	– 0.15 x 10^{-2}	– 71.3
0.10	B	0.257	0.248	+ 0.15 x 10^{-2}	– 71.4
0.15	F	0.262	0.247	– 0.23 x 10^{-2}	– 78.4
0.15	B	0.272	0.247	+ 0.23 x 10^{-2}	– 78.6
0.20	F	0.279	0.244	– 0.30 x 10^{-2}	– 90.2
0.20	B	0.290	0.244	+ 0.30 x 10^{-2}	– 90.4

(I) All the quantities are nondimensional with the scaling parameters being the same as in Table 1 of Chapter 1.

(II) The quantity ω_s is the dimensional lowest natural frequency for circumferential wave number n = 8.

(II) F denotes forward traveling wave (in the same direction as the sense of rotation of the shell) and B denotes backward traveling wave.

Table 2

Coefficients in the Amplitude Frequency Equation (13)

for a Clamped Circular Cylindrical Shell[I]

$(L/R = 1, \; \nu = 0.3, \; R/h = 100, \; n = 8)$

Ω/ω_s	Type of Wave	ω	m	g	γ_1
0	F,B	0.295	0.225	0	– 110.7
0.5	F	0.296	0.225	$- 0.80 \times 10^{-3}$	– 111.2
0.5	B	0.300	0.225	$+ 0.80 \times 10^{-3}$	– 111.9
0.10	F	0.304	0.224	$- 0.16 \times 10^{-2}$	– 115.5
0.10	B	0.311	0.224	$+ 0.16 \times 10^{-2}$	– 117.1
0.15	F	0.318	0.222	$- 0.24 \times 10^{-2}$	– 125.0
0.15	B	0.329	0.222	$+ 0.24 \times 10^{-2}$	– 128.0
0.20	F	0.237	0.220	$- 0.314 \times 10^{-2}$	– 141.2
0.20	B	0.351	0.220	$+ 0.314 \times 10^{-2}$	– 145.9

[I] All the footnotes of Table 1 apply to this Table as well.

CHAPTER III

NONLINEAR FLUTTER OF SHELLS OF REVOLUTION

INTRODUCTION

In this chapter we present an analysis of nonlinear flutter of shells
of revolution. A comprehensive summary of the related literature can be
found in the excellent reviews by Dowell [1,2] and Fung [3].

A number of experimental studies of flutter of cylindrical shells
were conducted by Olson and Fung [4,5], Stearman [6] and Stearman et. al [7].
Although there is not much disagreement on the qualitative aspects of flutter
between the observations and analytical results, the agreement on the quan-
titative features differs to a varying degree depending upon the approxima-
tions made on aerodynamic forces in the mathematical model. Most of the
comparisons have been made with the linearized theories of flutter, with the
main emphasis being placed on the flutter boundary which delineates, in the
space of relevant parameters, the region wherein flutter occurs from those
in which the shell is stable.

Linearized models of flutter of a circular cylindrical shell have been
studied by Miles [8], Dowell [9], Le Clerc [10], and Voss [11], among others.
Although the linear theory is adequate for the prediction of onset of flutter,
it does not provide any information on the amplitude of the fluttering
oscillations. For obtaining such results, analysis of the nonlinear problem
is necessary. Another important use of the results from nonlinear analysis
is that on their basis it can be ascertained as to whether or not there is a
possibility of the existence of fluttering motions at values of the relevant
parameters which are lower than the critical values predicted by linear theory.
Finally, for circular cylindrical shells, the linear theory fails to make any
distinction between the circumferentially traveling waves and standing waves
as modes of flutter. Thus, although some experimental results seem to suggest

that the flutter of a circular cylindrical shell is always associated with circumferentially traveling waves [5], the linear theory predicts that fluttering motion in the form of standing waves is equally likely to occur.

In order to alleviate the aforementioned limitations of linearized analysis, some analyses of nonlinear flutter of a cylindrical shell have also been conducted [12,13]. Evensen and Olson [12] have essentially used an assumed mode approach together with the Galerkin's method to analyze flutter of a shell using shallow-shell theory and first-order piston theory for aerodynamic forces. Although their solution does include both traveling wave and standing wave responses, the accuracy of the results is limited by the fact that the analysis does not include: (a) the nonlinear terms in the boundary conditions on axial stress-resultants for the case considered and (b) in-plane inertia terms in the equations of motion. Although one of these limitations has motivated an apparently more accurate analysis [13], yet in this solution, too, the nonlinear boundary conditions have been satisfied only in an averaged sense, and, in contrast to [12], no interaction between circumferentially traveling waves is considered.

In what follows is presented an analysis which takes into account the structural nonlinearities in a consistent manner. The results obtained can form a basis for a computational algorithm for analysis of flutter of shells of revolution along the lines of the numerical schemes for nonlinear oscillations described in the previous chapters.

ANALYSIS

As for the nonlinear oscillations problem, the analysis is most easily carried out by writing the equilibrium equations in the Koiter's notation [14] in the form

$$\omega^2 M_{11}(\ddot{q}, \delta q) + \omega C_{11}(\dot{q}, \delta q) + \lambda A_{11}(q, \delta q)$$
$$+ P_{11}(q, \delta q) + P_{21}(q, \delta q) + P_{31}(q, \delta q) = 0. \qquad (1)$$

The vector q in (1) represents the three displacements associated with the shell, ω is the frequency of oscillation, superposed dot denotes derivative with respect to time nondimensionalized by the frequency of oscillation and the symmetric functional M_{11} represents the inertia terms. Further, the terms $P_{i1}(i = 1,2,3)$ represent the first variation of homogeneous functionals of degree $(i + 1)$, and they, of course, arise from the structural stiffness, with the nonlinearities being a consequence of the quadratic strain-displacement relations. Finally, the functionals C_{11} and A_{11} represents, respectively, the damping and the nonconservative forces on the structure. The quantity λ is a "load" parameter essentially representing, in case of aerodynamic flutter, the speed of the fluid flowing over the shell. Although, in general, the load parameter can appear nonlinearly as well, and in the other terms (particularly in the damping term in the equilibrium equation) it is sufficient for describing the basic methodology to assume the particular form (1).

It is also noted here that for shells of revolution each of the components of q is periodic with respect to the angular coordinate: this is a result of the invariance of the equilibrium equations with respect to angular rotation and it leads to an interesting problem of interaction between circumferentially traveling waves in nonlinear flutter.

Due to the scaling of the time coordinate in (1) by the frequency of oscillation ω, we seek time periodic solutions of this equation with a period 2π. The equation has a trivial solution for all values of ω, λ. The condition for the existence of bifurcation points on the trivial solution branch leads to the linear eigenvalue problem

$$-\omega_o^2 M_{11}(y, \delta q) + i\omega_o C_{11}(y, \delta q) + \lambda_o A_{11}(y, \delta q) + P_{11}(y, \delta q) = 0, \tag{2}$$

where ω_o is the frequency at the onset of flutter and λ_o is the associated load parameter.

Equation (2) is derived from (1) on the assumption that the solution at the incipient loss of instability is of the form

$$y\, e^{it}$$

so that the functions

$$\bar{y}\, e^{-it}, \quad y\, e^{-it}, \quad \bar{y}\, e^{it}$$

are such solutions as well. Due to rotational symmetry the flutter mode y is of the form

$$y = \hat{y}\, e^{in\theta} \tag{3}$$

where $\hat{y}$ depends upon the axial coordinate only.

It is convenient to also denote here the solution of the problem adjoint to (2) by z, i.e., z satisfies

$$-\omega_o^2 M_{11}(z, \delta q) - i\omega_o C_{11}^*(z, \delta q)$$

$$+ \lambda_o A_{11}^*(z, \delta q) + P_{11}(z, \delta q) = 0 \tag{4}$$

where C_{11}^* and A_{11}^* represent the adjoints of the functionals C_{11} and A_{11}, respectively; thus, for any displacements q_1, q_2,

$$\bar{C}_{11}(q_1, \bar{q}_2) = C_{11}^*(q_2, q_1).$$

For the values of (λ, ω) near (λ_o, ω_o), the solution of (1) is written as

$$q = (\alpha\, y\, e^{it} + c.c.) + (\beta\, \bar{y}\, e^{it} + c.c.) + W, \tag{5}$$

where w is as yet an unknown function satisfying an orthogonality condition

$$\int_{c}^{2\pi} e^{-it} T_{11}(W,\bar{z})dt = 0, \tag{6a}$$

$$\int_{0}^{2\pi} e^{-it} T_{11}(W,z)dt = 0, \tag{6b}$$

where T_{11} is the first variation of an arbitrarily positive definite quadratic functional. It may be noted here that the choice of T_{11} together with the orthogonality conditions (6) fixes the amplitude a and β uniquely, i.e.

$$a = \int_{0}^{2\pi} e^{-it} T_{11}(q,\bar{z})dt \tag{7a}$$

$$\beta = \int_{0}^{2\pi} e^{-it} T_{11}(q,z)dt \tag{7b}$$

where we have assumed that the solutions of (2,4) have been normalized so that

$$\int_{0}^{2\pi} T_{11}(y,\bar{z})dt = 1$$

$$\int_{0}^{2\pi} T_{11}(\bar{y},z)dt = 1.$$

Just as the solution has been decomposed in (5), we also decompose equation (1) into three equivalent ones:

$$L_{11}(W,\delta q) = R(q,\delta q) - \{e^{it}T_{11}(y,\delta q)\int_{0}^{2\pi} e^{-it}R(q,\bar{z})dt + c.c.\}$$

$$-\{e^{it}T_{11}(\bar{y}, q)\int_{0}^{2\pi} e^{-it}R(q,z)dt + c.c\}, \tag{8a}$$

$$\int_{0}^{2\pi} e^{-it}R(q,\bar{z})dt = 0, \tag{8b}$$

$$\int_{0}^{2\pi} e^{-it}R(q,z)dt = 0, \tag{8c}$$

where

$$L_{11}(W,\delta q) \equiv \omega_o^2 M_{11}(W,\delta q) + \omega_o C_1(\dot{W},\delta q) + \lambda_o A_{11}(W,\delta q) + P_{11}(W,\delta q), \tag{9a}$$

and

$$R(q,\delta q) = \omega_o^2 M_{11}(\ddot{q},\delta q) + \omega C_1(\dot{q},\delta q) + \lambda A_{11}(q,\delta q) + P_{11}(q,\delta q) + P_{21}(q,\delta q)$$

$$+ P_{31}(q,\delta q) - L_{11}(W,\delta q). \tag{9b}$$

It can be verified easily that there are no secular terms in (8a), i.e., on substitution of $\delta q = ze^{it}$ or $\bar{z}e^{it}$ the equation (8a) can be shown to be identically satisfied. As a result of this, we can solve (8a) in an asymptotic series in α, β, $(\omega - \omega_o)$ and $(\lambda - \lambda_o)$. The solution adequate for our purpose is found to be

$$W = (\alpha^2 w_{11}^{(2)} e^{2it} + c.c.) + \alpha\bar{\alpha}w_{11}^{(o)} + (\beta^2 w_{22}^{(2)} e^{2it} + c.c.) + \beta\bar{\beta}w_{22}^{(o)}$$

$$+ (\alpha\beta w_{12}^{(2)} e^{2it} + c.c.) + (\alpha\bar{\beta}w_{12}^{(o)} + c.c.) + 0(\alpha^3,\beta^3,(\omega-\omega_o),(\lambda-\lambda_o)). \tag{10}$$

where the participating mode $w_{11}^{(2)}$, $w_{11}^{(o)}$, etc., satisfy the linear, time-independent problems:

$$Q_{11}(w_{11}^{(2)},\delta q) + P_{21}(v,\delta q) = 0, \tag{11a}$$

$$Q_{11}(w_{12}^{(2)},\delta q) + P_{111}(y,\bar{y},\delta q) = 0, \tag{11b}$$

$$\lambda_o A_{11}(w_{11}^{(1)},\delta q) + P_{11}(w_{11}^{(1)},\delta q) + P_{111}(y,\bar{y},\delta q) = 0, \tag{11c}$$

$$\lambda_o A_{11}(w_{12}^{(o)},\delta q) + P_{11}(w_{12}^{(o)},\delta q) + 2P_{21}(y,\delta q) = 0, \tag{11d}$$

and where

$$Q_{11}(q,\delta q) \equiv - 4\omega_o^2 M_{11}(q,\delta q) + 2i\omega_o C_{11}(q,\delta q)$$

$$+ \lambda_o A_{11}(q,\delta q) + P_{11}(q,\delta q). \tag{12}$$

Once the solutions of (11) have been obtained, the other participating modes
in (10) are calculated simply from the relations

$$w_{22}{}^{(2)} = \bar{w}_{11}{}^{(2)}, \quad w_{22}{}^{(o)} = w_{11}{}^{(o)},$$

which are obtained by examining the nonhomogeneous terms in equations for
$w_{22}{}^{(2)}$, $w_{22}{}^{(o)}$ etc.

With (8a) having been solved in the form of the series (10), we can sub-
stitute the result into (8b,c) to obtain the nonlinear equations relating
the flutter amplitudes, frequency and the load parameters. The equations
turn out to be

$$-m(\omega^2 - \omega_o{}^2)\alpha + ic(\omega - \omega_o)\alpha + (\lambda - \lambda_o)a\alpha + \gamma_1\alpha^2\bar{\alpha} + \gamma_2\alpha\beta\bar{\beta} = 0, \tag{13a}$$

$$-m(\omega^2 - \omega_o{}^2)\beta + ic(\omega - \omega_o)\beta + (\lambda - \lambda_o)a\beta + \gamma_1\beta^2\bar{\beta} + \gamma_2\beta\alpha\bar{\alpha} = 0, \tag{13b}$$

where

$$m = M_{11}(y,\bar{z}), \tag{14a}$$

$$c = C_{11}(y,\bar{z}), \tag{14b}$$

$$a = A_{11}(y,\bar{z}), \tag{14c}$$

and

$$\gamma_1 = P_{111}(w_{11}{}^{(2)}, \bar{y},\bar{z}) + P_{111}(w_{11}{}^{(1)},y,\bar{z}) + P_{211}(y,\bar{y},\bar{z}), \tag{15a}$$

$$\gamma_2 = 2P_{21}(y,\bar{w}_{12}{}^{(o)}) + P_{111}(w_{12}{}^{(2)},y,\bar{z})$$

$$+ P_{111}(w_{11}{}^{(o)},y,\bar{z}) + 2P_{211}(y,\bar{y},\bar{z}). \tag{15b}$$

Thus equations (13a,b) are the final results of the analysis. From these
equations one can obtain both the circumferentially traveling wave response
and the standing wave response, as well as the response that arises due to

the interaction between them. Although these equations are four equations in six variables — ?, and the complex amplitudes a and β — it can be shown that they admit solutions such that the phases of the (complex) amplitudes are arbitrary. Thus the equations (13) can be used for the calculation of the load parameters and frequency of harmonic flutter as functions of the amplitudes of the two traveling waves.

We end this section by emphasizing that the results presented above are asymptotic in nature, and, therefore, they are valid for sufficiently small but nonvanishing amplitudes. (We note, also, that the linear theory of flutter is valid only for the limiting case of vanishing amplitude.)

CONCLUDING REMARKS

We have presented an algorithm whose implementation within the framework of the finite element methodology can provide a useful means for obtaining the effect of structural nonlinearities on the flutter of shells of revolution.

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